Since, for any integral values of n, the numbers, $(n-1)$, $(n-2)$, $\cdots$, $(n-p)$, are p consecutive integers, one and only one of them is divisible by p.

Suppose $(n-k)$ is divisible by p, *i. e.*, $n \equiv k \pmod{p}$. Then the coefficient of A_k is p and all the other coefficients are zero.

Hence, when $n \equiv k \pmod{p}$, $f = A_k$ as required.

244. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

Determine the rational value of x that will render $x^2 + px + q$ a perfect square. What value of x will render $x^2 - 7x + 2$ a perfect square?

Solution by Harold T. Davis, Colorado Springs, Colorado.

Let $x^2 + px + q = y^2$. Then $x^2 + px + (p^2/4) + q - (p^2/4) - y^2 = 0$, or $(2x+p)^2 - (2y)^2 = p^2 - 4q$.

(1) Let $2x + p = z$ and $2y = w$. Then $z^2 - w^2 = p^2 - 4q$; or, if $4q > p^2$, $w^2 - z^2 = 4q - p^2$. Let a and b be complementary factors of $p^2 - 4q$. Then $z + w = a$ and $z - w = b$. Whence $z = (a+b)/2$ and $w = (a-b)/2$. Substituting these values in equations (1), we have

$$x = \frac{a+b-2p}{4} \text{ and } y = \frac{a-b}{4}.$$

Example. $x^2 - 7x + 2 = y^2$. Here $p^2 - 4q = 41$, the complementary factors of which are 41 and 1. Hence,

$$x = \frac{41+1+14}{4} = 14 \quad \text{and} \quad y = \frac{41-1}{4} = 10.$$

A complete discussion of the solution of the general equation of the second degree in two variables is given in Chrystal's *Algebra*, Part II, page 458.

Also solved by Horace Olson, Norman Anning, H. N. Carleton, O. S. Adams, J. A. Colson, J. L. Riley, N. Pandya and J. H. Weaver.

QUESTIONS AND DISCUSSIONS.

Send all communications to U. G. Mitchell, University of Kansas, Lawrence.

REPLIES.

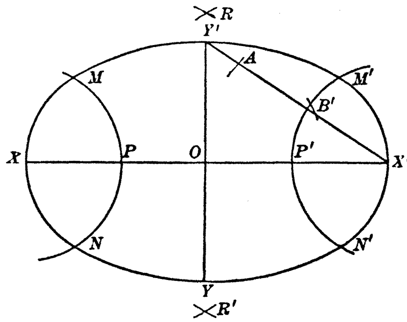
33. Under what conditions or to what extent is Mr. Iwerson's construction, given below, a useful or practical approximation to a true ellipse? What criterion can be given to measure definitely the degree of approximation?

Mr. Iwerson's approximate construction for an ellipse by ruler and compasses alone, having given the axes, was given in the November, 1916, issue of the Monthly, pp. 354, 355. The following corrections should be made: In the last two lines on p. 354, Ox should be OY, and Oy should be OX.

Note. In the February issue of the Monthly (pp. 90–92), we published a reply to this question by Professor Capron, of the U. S. Naval Academy. Before the February issue had come from the press, Professor Howland, of Wesleyan University, sent in the reply printed below. These two discussions are, accordingly, independent and from entirely different points of view. Professor Capron took as his primary measure of approximation the proportional errors in the radii of curvature at important points. Professor Howland has taken as his measure of approximation the ratio of the distance between the true and constructed curves (measured vertically or normally) to the semi-major axis. The two discussions seem to overlap in but one place. What Professor Capron has called the proportional error in the length of the minor axis and designated as E_1, corresponds to a maximum value of Professor Howland's relative divergence, d_2/a, which for some eccentricities occurs at $x = 0$. There is some slight difference in the formulas since in the one case the error is given relative to the semi-minor and in the other case relative to the semi-major axis.

The behavior of the arc MM', as brought out in the discussion below, is decidedly interesting. One would certainly not imagine *a priori* that the arc would cut the ellipse in four real distinct points.—U. G. M.

REPLY BY LEROY A. HOWLAND, Wesleyan University, Middletown, Conn.



In the figure, if the axes of the ellipse be taken as coördinate axes, the coördinates of the lettered points are readily found as follows:

$$X',\ (a, 0);\quad Y',\ (0, b);\quad B,\ \left(\frac{2ab^2}{a^2+b^2},\ \frac{a^2b-b^3}{a^2+b^2}\right);$$

$$A,\ \left(a-\frac{a^2}{\sqrt{a^2+b^2}},\ \frac{ab}{\sqrt{a^2+b^2}}\right);\quad M',\ \left(\frac{a+k}{2},\ \frac{(a-k)\sqrt{3}}{2}\right),$$

where

$$k=\frac{a^2-b^2}{\sqrt{a^2+b^2}};$$

$$P',\ (k, 0);\quad R',\ (0, -k\sqrt{3});\quad AB'=a-k;\quad MM'=a+k.$$

Since $\begin{vmatrix} \frac{a+k}{2}, & \frac{(a-k)\sqrt{3}}{2}, & 1 \\ k & 0 & 1 \\ 0 & -k\sqrt{3} & 1 \end{vmatrix} = 0$, M', P' and R' lie on a straight line

and since R' and P' are the centers of curvature of the arcs MM' and $M'N'$ respectively, the reason becomes evident why these arcs meet (in M') at an angle of 0°.

The equation of the ellipse, the circle about P', and the circle about R' are, respectively,

(1) $$x^2/a^2+y^2/b^2=1,$$

(2) $$(x-k)^2+y^2=(a-k)^2,$$

(3) $$x^2+(y+k\sqrt{3})^2=(a+k)^2.$$

We are, therefore, interested in the three arcs

$$(1') \qquad y = \frac{b}{a}\sqrt{a^2 - x^2},$$

$$(2') \qquad y = \sqrt{(a-k)^2 - (x-k)^2}, \qquad \frac{a+k}{2} \leqq x \leqq a,$$

$$(3') \qquad y = \sqrt{(a+k)^2 - x^2} - k\sqrt{3}, \qquad -\frac{a+k}{2} \leqq x \leqq \frac{a+k}{2}.$$

The distances of the circular arcs from the ellipse are given by

$$d_1 = \sqrt{(a-k)^2 - (x-k)^2} - \frac{b}{a}\sqrt{a^2 - x^2},$$

$$d_2 = \sqrt{(a+k)^2 - x^2} - k\sqrt{3} - \frac{b}{a}\sqrt{a^2 - x^2}.$$

If we put $b/a = \lambda$, $x = ax'$, then

$$k = \frac{1 - \lambda^2}{\sqrt{1 + \lambda^2}}\,a,$$

and the ratios d_1/a and d_2/a, which we may call the relative divergences of the circular arcs, are functions of λ and x'.

If $a = b$ the construction is obviously exact, hence we shall assume $a > b$ or $\lambda < 1$.

Within its interval d_1 can easily be shown to be always greater than zero. To find its maximum, we have

$$\frac{d(d_1)}{dx} = -\frac{x - k}{\sqrt{(a-k)^2 - (x-k)^2}} + \frac{bx}{a\sqrt{a^2 - x^2}}.$$

The critical values are found to be $x = a$, which gives a minimum $d_1 \doteq 0$, and the roots of the cubic

$$(a^2 - b^2)x^3 + (a^2 - b^2)(a - 2k)x^2 + a^2k(k - 2a)x + a^3k^2 = 0.$$

This cubic has three real roots, one in each of the intervals

$$(-\infty, 0), \left(0, \frac{a+k}{2}\right), \quad \text{and} \quad \left(\frac{a+k}{2}, a\right).$$

There is, therefore, always one and only one critical value between $(a + k)/2$ and a. This gives a maximum value of d_1.

A discussion of d_2 can be made more complete, but is less simple, for the reason that in its interval it changes sign, in some cases as many as four times. To show this, we will find the intersections of

$$x^2/a^2 + y^2/b^2 = 1$$

and

$$x^2 + (y + k\sqrt{3})^2 = (a + k)^2.$$

Eliminating x^2 we have

$$(a^2 + b^2)y^2 - 2\sqrt{3}b^2\sqrt{a^2 + b^2}y + 2b^2(a\sqrt{a^2 + b^2} - a^2 + b^2) = 0,$$

whence

$$y = \frac{b^2\sqrt{3} \pm b(\sqrt{a^2 + b^2} - a)}{\sqrt{a^2 + b^2}}.$$

Both of these values of y can be shown to be greater than the ordinate of M' and hence all real intersections of these two curves occur along the arc under discussion. The corresponding values of x are given by the equations

$$x^2 = \frac{a^2(a - b\sqrt{3})[2\sqrt{a^2 + b^2} - a + b\sqrt{3}]}{a^2 + b^2},$$

$$x^2 = \frac{a^2(a + b\sqrt{3})[2\sqrt{a^2 + b^2} - a - b\sqrt{3}]}{a^2 + b^2}.$$

The expressions in brackets are always positive. Hence we have three distinct cases:

(1) $a < b\sqrt{3}$, two real distinct intersections;

(2) $a = b\sqrt{3}$, two real distinct and two coincident;

(3) $a > b\sqrt{3}$, four real distinct.

In case (1), d_2 becomes zero twice and changes sign twice. In case (2), d_2 becomes zero three times and changes sign twice. In case (3), d_2 becomes zero four times and changes sign four times. To find the maxima of d_2, we have

$$\frac{d(d_2)}{dx} = \frac{-x}{\sqrt{(a + k)^2 - x^2}} + \frac{bx}{a\sqrt{a^2 - x^2}}.$$

The critical values are

$$x_1 = -\sqrt{\frac{a^4 - b^2(a + k)^2}{a^2 - b^2}}, \qquad x_2 = 0, \qquad x_3 = \sqrt{\frac{a^4 - b^2(a + k)^2}{a^2 - b^2}},$$

$$\frac{d^2(d_2)}{dx^2} = -\frac{(a + k)^2}{[(a + k)^2 - x^2]^{3/2}} + \frac{ab}{(a^2 - x^2)^{3/2}},$$

$$\left.\frac{d^2(d_2)}{dx^2}\right|_{x=0} = -\frac{1}{a + k} + \frac{b}{a^2} = \frac{-a^2 + b(a + k)}{a^2(a + k)}.$$

This is evidently positive and gives a minimum so long as $a^2 < b(a + k)$, that is, so long as the critical points x_1 and x_3 are imaginary. Since d_2 is now negative, this minimum is a maximum of its absolute value. When $a^2 = b(a + k)$, the three critical points coincide at $x = 0$. It is found in this case that

$$\frac{d^2(d_2)}{dx^2} = \frac{d^3(d_2)}{dx^3} = 0,$$

but

$$\frac{d^4(d_2)}{dx^4} > 0,$$

and we still have a minimum d_2 or maximum $| d_2 |$. For $a^2 > b(a+k)$, $x = 0$ gives a maximum d_2, which has no interest for us until d_2 becomes positive for $x = 0$, *i. e.*, until $a > \sqrt{3}b$. This value is $a - b + (1 - \sqrt{3})k$ or, introducing $b/a = \lambda$,

$$\left|\frac{d_2}{a}\right|_{x=0} = 1 - \lambda + (1 - \sqrt{3})\frac{1-\lambda^2}{\sqrt{1+\lambda^2}}.$$

This function of λ can be shown to have a negative derivative for $\lambda < 1/\sqrt{3}$, hence this maximum divergence increases as λ decreases and approaches as a limit

$$2 - \sqrt{3} \doteq 0.268\cdots.$$

We also find that at x_1 and x_3, as soon as they become real, we have maxima for $| d_2 |$ equal to

$$k\sqrt{3} - \frac{\sqrt{a^2-b^2}\sqrt{(a+k)^2-a^2}}{a}.$$

or

$$\left|\frac{d_2}{a}\right|_{x=x_1} = \frac{1-\lambda^2}{\sqrt{1+\lambda^2}}[\sqrt{3} - \sqrt{2\sqrt{1+\lambda^2}+1-\lambda^2}].$$

This function of λ is found to be an increasing function[1] for $0 < \lambda < 0.72\cdots$. Hence as λ decreases, $| d_2/a |$ decreases and approaches the limit 0.

The following typical cases indicate the degree of approximation of the construction:

λ	critical points	max. of $\left\lvert\frac{d_1}{a}\right\rvert$	max. of $\left\lvert\frac{d_2}{a}\right\rvert$
$\frac{3}{4}$	$x = 0.858a$	0.0222	——
	$x = 0$	——	0.0062
	$x = \pm\frac{\sqrt{-23}}{20}a$	——	——
$\frac{1}{\sqrt{3}}$	$x = 0.905a$	0.0217	——
	$x = 0$	——	——
	$x = \pm 0.506a$	——	0.0040
$\frac{5}{12}$	$x = 0.942a$	0.0156	——
	$x = 0$	——	0.0249
	$x = \pm 0.747a$	——	0.0015

[1] $a^2 > b(a+k)$, $1 > \lambda\left(1 + \frac{1-\lambda^2}{\sqrt{1+\lambda^2}}\right)$, $\sqrt{1+\lambda^2} > \lambda(1+\lambda)$, $\lambda^4 + 2\lambda^3 - 1 < 0$ or $\lambda < 0.72$.

The above discussion shows the vertical divergence between the true and the constructed ellipse. A better measure of the approximation, however, is the divergence along radii of the circular arcs.

If we transform our equation of the ellipse to axes through the point P' and then change to polar coördinates, we have:

$$\rho^2[a^2 - (a^2 - b^2)\cos^2\theta] + 2b^2k\rho\cos\theta + b^2(k^2 - a^2) = 0$$

or, introducing λ and letting $\cos\theta = t$,

$$\rho^2(1+\lambda^2)[1-(1-\lambda^2)t^2] + 2\lambda^2(1-\lambda^2)\sqrt{1+\lambda^2}\,at\rho + \lambda^2(\lambda^4 - 3\lambda^2)a^2 = 0.$$

This gives for the ellipse

$$\frac{\rho}{a} = -\frac{\lambda^2}{\sqrt{1+\lambda^2}}\left[\frac{(1-\lambda^2)t - \sqrt{3-\lambda^2-2(1-\lambda^2)t^2}}{1-(1-\lambda^2)t^2}\right].$$

For the circle about P' we have

$$\rho' = a - k = a\left(1 - \frac{1-\lambda^2}{\sqrt{1+\lambda^2}}\right)$$

or

$$\frac{\rho'}{a} = 1 - \frac{1-\lambda^2}{\sqrt{1+\lambda^2}}.$$

$\dfrac{\rho' - \rho}{a} = n_1$ we may call the relative normal divergence. The critical values of θ are found from the equation

$$\sqrt{1-t^2}\left\{[1-(1-\lambda^2)t^2]\left[1-\lambda^2 - \frac{2(1-\lambda^2)t}{\sqrt{3-\lambda^2-2(1-\lambda^2)t^2}}\right] \right.$$
$$\left. + 2(1-\lambda^2)t[(1-\lambda^2)t - \sqrt{3-\lambda^2-2(1-\lambda^2)t^2}]\right\} = 0.$$

$t = 1$ or $\theta = 0$ gives us the minimum $n_1 = 0$. Equating the other factor to zero and simplifying, we obtain

$$2(1-\lambda^2)^2t^6 - (1-\lambda^2)(5-\lambda^2)t^4 + 2(2-\lambda^2)t^2 - 1 = 0$$

or

$$[(1-\lambda^2)t^2 - 1]^2[2t^2 - 1] = 0.$$

We have then the rather remarkable result that the value of θ giving a maximum n_1 is independent of λ and is always 45°. This maximum is

$$n_1 = 1 - \frac{1 + (\sqrt{2}-1)\lambda^2}{\sqrt{1+\lambda^2}},$$

and

$$\frac{d(\max n_1)}{d\lambda} = \frac{(\sqrt{2}-1)\lambda}{(1+\lambda^2)^{3/2}}[\sqrt{2} - 1 - \lambda^2].$$

As λ decreases, therefore, max n_1 increases until $\lambda^2 = \sqrt{2} - 1$, when it has the value $0.0148\cdots$. It then decreases, approaching 0 with λ.

In similar manner, the equation of the ellipse, referred to R' as pole and a horizontal through R' as initial line, is

$$\rho/a = \frac{1}{\sqrt{1+\lambda^2}}\left[\frac{\sqrt{3}(1-\lambda^2)s + \lambda\sqrt{2(1-\lambda^2)(2-\lambda^2)s^2 - (3-\lambda^2)(1-2\lambda^2)}}{(1-\lambda^2)s^2+\lambda^2}\right],$$

where $s = \sin\theta$. For the circle $\rho'/a = 1 + \dfrac{1-\lambda^2}{\sqrt{1+\lambda^2}}$

$|(\rho' - \rho)/a| = n_2$ will be the relative normal divergence for these arcs. Critical values of θ are found from an equation which in simplified form becomes

$$\sqrt{1-s^2}\,[(1-\lambda^2)s^2+\lambda^2]^2\,[2(2-\lambda^2)s^2-3] = 0.$$

For $s = 1$, $\theta = 90°$, we have the vertical divergence discussed previously. The other real critical value is given by

$$s = +\sqrt{\frac{3}{2(2-\lambda^2)}}, \qquad \lambda^2 \lesseqgtr \tfrac{1}{2}.$$

The corresponding maximum is

$$|n_2| = \frac{\sqrt{2(2-\lambda^2)}}{\sqrt{1+\lambda^2}} - \frac{1-\lambda^2}{\sqrt{1+\lambda^2}} - 1.$$

The three typical cases, computed above, give the following results:

λ	max $\lvert n_1\rvert$ $s^2 = \frac{1}{2}$	max $\lvert n_2\rvert$	
		$s = 1$	$s = +\sqrt{\frac{3}{2(2-\lambda^2)}}$
$\frac{3}{4}$	0.0137	0.0062	——
$\frac{1}{\sqrt{3}}$	0.0145	0.0000	0.0038
$\frac{5}{12}$	0.0106	0.0249	0.0014

DISCUSSIONS.

Relating to the Indeterminate Form 0/0.

By M. O. Tripp, Olivet College, Olivet, Michigan.

In the May, 1916, number of the Monthly (Vol. XXIII, p. 180) Professor J. W. Nicholson considers the equation

$$y = \frac{x^2 - a^2}{x - a}, \tag{1}$$